
SEQUENCES

□ INTRODUCTION TO SEQUENCES

A **sequence** is a listing of numbers in a particular order. (Actually, a sequence can be a listing of almost anything you can imagine, like baseball games, but for the first

couple of Algebra courses we'll assume sequences of numbers.) For example: 1, 3, 8, 5 is a **finite** sequence of four terms, and is different from the finite sequence 3, 5, 1, 8 (even though each sequence consists of the same four numbers).

A sequence can also be **infinite**; an example would be

$$0, 2, 4, 6, 8, 10, 12, \dots,$$

which represents all the **even** whole numbers.

Since order is the key idea behind a sequence, a sequence has a first term, a second term, a third term, and so on. For instance, the sequence (notice the ellipsis)

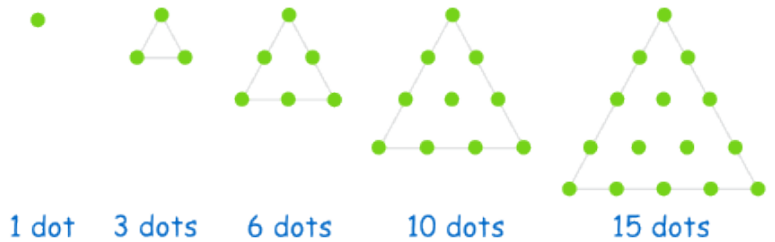
$$5, 8, 11, 14, \dots, 29$$

is a finite sequence consisting of nine terms:

$$5, 8, 11, 14, 17, 20, 23, 26, 29$$

The second term is 8, the fifth term is 17, and the ninth term is 29.

Our goal is to look at the first few terms of a given sequence, and then try to determine the value of a term farther out in the sequence.



The three dots “...” is an English punctuation symbol called an *ellipsis*, and denotes an *omission* of something — or in this case it means “and so on, forever.”

Homework

1. Determine whether the sequence is finite or infinite:
 - a. 2, 3, 4, 5, ...
 - b. 5, 7, 9, ... 1001
 - c. 7, 7, 7, 7, ...
 - d. π, π^2, π^3, π^4

2.
 - a. What is the 7th term of the sequence 4, 8, 3, 0, 0, 9, 13, -1, π ?
 - b. Are the sequences 1, 2, 3, 4 and 4, 3, 2, 1 the same sequence?

□ **EXAMPLES OF SEQUENCES**

EXAMPLE 1: Consider the infinite sequence

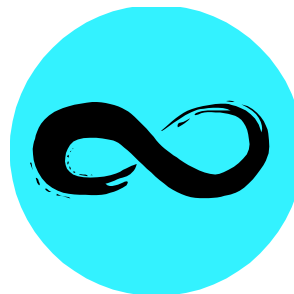
4, 8, 12, 16, 20, ...

**The 1st term is 4, the 2nd term is 8, and so on.
What's the 10th term?**

Solution: Here's an approach: Find the pattern among the existing terms, extend that pattern, and then predict the 10th term. Each term of the sequence appears to be four more than the preceding term. Continuing this pattern, it appears that the first 10 terms of the sequence are

4, 8, 12, 16, 20, 24, 28, 32, 36, 40.

Hence, **the 10th term is 40**, and we've accomplished our goal.



EXAMPLE 2: Consider the sequence in Example 1 again:
4, 8, 12, 16, 20, . . . This time find the 900th term.

Solution: Would you like to list 900 terms to see what the 900th term is? I didn't think so! Let's attack this problem from a different perspective. Here's the idea in a nutshell: *Instead of seeing how each term comes from the ones before it, let's see how each term comes from its position in the sequence.* We'll let n represent the position.

n	1	2	3	4	5
Term	4	8	12	16	20

Let's determine each term as a *function* of its position, n .

This may not be obvious at first, but notice that each term is 4 times its position. For example, the term 12 is 4 times its position of 3. The term 20 is 4 times its position of 5. As a formula, we can write

$$nth \text{ term} = 4n$$

Answering the original question is now a snap. The 900th term is given by the expression $4(900)$, which is **3600**, and we're done.

EXAMPLE 3: Find the 1000th term of the sequence
4, 7, 10, 13, 16, . . .

Solution: As in Example 2, let's construct a chart showing each term of the sequence as a function of its position:

n	1	2	3	4	5
Term	4	7	10	13	16

A little analysis (a phrase that means "it's not easy!") shows that each term is 1 more than 3 times its position. For example, the term 16 is 1 more than 3 times its position of 5. In general,

$$nth \text{ term} = 3n + 1$$

Therefore, the 1000th term is $3(1000) + 1 = \mathbf{3001}$.

EXAMPLE 4: Find the 1000th term of 1, 4, 9, 16, 25, 36, . . .

Solution: Listing our data in a nice chart gives:

n	1	2	3	4	5	6
Term	1	4	9	16	25	36

This example doesn't have the same kind of formula that's in the previous examples, since the terms are not growing at a constant rate — they're getting big very quickly. So let's look at it in a somewhat different way.

Notice that the term 25 is the square of its position $n = 5$. In fact, every term of the sequence is the square of its position (you verify this, for if there's just one place where our theory fails, then we have the wrong formula). We can write this as

$$nth \text{ term} = n^2$$

And now it's just arithmetic. The 1000th term is $1000^2 = \mathbf{1,000,000}$.

EXAMPLE 5: Find the 500th term of 2, 5, 10, 17, 26, 37, . . .

Solution: Let's set up a chart.

n	1	2	3	4	5	6
Term	2	5	10	17	26	37

The trick here is to notice that each term of this sequence is one more than the square of the position. For instance, the term 26 is 1 more than the square of its position of 5. Thus, our formula here needs to be

$$nth \text{ term} = n^2 + 1$$

The 500th term is therefore $500^2 + 1 = \mathbf{250,001}$.

EXAMPLE 6: Find the 25th term of 2, 4, 8, 16, 32, 64, 128, . . .

Solution:

n	1	2	3	4	5	6	7
Term	2	4	8	16	32	64	128

This one is also kind of obscure. But notice that each term is twice the previous term. This leads us to the realization that each term of the sequence can be found by raising 2 to the power of its position. For instance, the 5th term is $2^5 = 32$. Check out the first term: $2^1 = 2$. Also, the 7th term, which is 128, is indeed 2^7 . Our summary is therefore

$$nth \text{ term} = 2^n$$

The 25th term must be $2^{25} = 33,554,432$.

EXAMPLE 7: Consider the infinite sequence

-1, 1, 5, 13, 29, 61, 125, . . .

Find the 20th term.

Solution:

n	1	2	3	4	5	6	7
Term	-1	1	5	13	29	61	125

Think powers of 2 (as in the previous example). But this time notice that each of the terms of the sequence is 3 less than 2 raised to the power of its position. As an example, the 6th term (61) is 3 less than 2^6 (64). In general,

$$nth \text{ term} = 2^n - 3.$$

In conclusion, the 20th term is $2^{20} - 3 = 1,048,573$.

EXAMPLE 8: Find the 125th term of 1, 8, 27, 64, 125, 216, . . .

Solution: Don't be upset if you don't see a pattern here. I wouldn't be able to see it either if I hadn't made up the problem myself. This is the sequence of perfect cubes:

$$1^3, 2^3, 3^3, 4^3, 5^3, 6^3, \dots$$

Thus, the formula for the n th term is

$$n\text{th term} = n^3$$

We therefore have a 125th term of $125^3 = \mathbf{1,953,125}$.

EXAMPLE 9: Find the 60th term of 0, 7, 26, 63, 124, 215, . . .

Solution: Comparing this sequence with the sequence of perfect cubes in Example 8, we see that each term is 1 less than a perfect cube, so the formula must be

$$n\text{th term} = n^3 - 1$$

Letting $n = 60$, we calculate the 60th term to be **215,999**.

Homework

3. Find the 2000th term of 7, 12, 17, 22, 27, . . .
4. Find the 100th term of -1, 2, 7, 14, 23, 34, . . .
5. Find the 25th term of 5, 7, 11, 19, 35, 67, . . .
6. Find the 57th term of -1, 6, 25, 62, 123, . . .
7. Find the 7354th term of 6, 9, 14, 21, 30, 41, 54, . . .
8. Find the 3000th term of 13, 17, 21, 25, 29, . . .
9. Find the 123rd term of 4, 11, 30, 67, 128, . . .

**“Life does
not need to
be changed.
Only your
intent and
actions do.”**

*Swami
Rama*

10. Find the 30th term of $-3, -1, 3, 11, 27, 59, \dots$
11. Find the 62nd term of $4, 7, 10, 13, 16, \dots$
12. Find the 81st term of $4, 7, 12, 19, 28, 39, \dots$
13. Find the 71st term of $-2, 5, 24, 61, 122, 213, \dots$
14. Find the 87th term of $0, 7, 26, 63, 124, 215, \dots$
15. Find the 67th term of $-3, 0, 5, 12, 21, 32, \dots$
16. Find the 64th term of $2, 5, 10, 17, 26, 37, \dots$

Review Problems

17. Find the 32nd term of $5, 7, 11, 19, 35, 67, 131, \dots$
18. Find the 200th term of $10, 13, 16, 19, \dots$
19. Find the 100th term of $4, 7, 12, 19, 28, \dots$
20. Find the 30th term of $1, 3, 7, 15, 31, 63, \dots$
21. Find the 100th term of $3, 10, 29, 66, 127, \dots$
22. True/False:
 - a. The sequence $1, 2, 3$ is the same sequence as $3, 2, 1$.
 - b. It's possible that the terms of a sequence are all the same.
 - c. The 2000th term of $20, 23, 26, 29, \dots$ is 6017.
 - d. The 100th term of $11, 14, 19, 26, 35, \dots$ is 10,000.
 - e. The 100th term of $-9, -2, 17, 54, 115, \dots$ is 999,990.
 - f. The 30th term of $10, 12, 16, 24, 40, 72, \dots$ is 1,073,741,832.

Solutions

1. a. Infinite b. Finite c. Infinite d. Finite
2. a. 13 b. No 3. 10,002 4. 9,998 5. 33,554,435
6. 185,191 7. 54,081,321 8. 12,009 9. 1,860,870
10. 1,073,741,819 11. 187 12. 6564 13. 357,908
14. 658,502 15. 4485 16. 4097 17. 4,294,967,299
18. 607 19. 10,003 20. 1,073,741,823
21. 1,000,002
22. a. F b. T c. T d. F e. T f. T

□ ***TO ∞ AND BEYOND***

“I think and think for months and years. Ninety-nine times, the conclusion is false. The hundredth time I am right.”

– Albert Einstein